ee364m Exercise Set 1

Due date: January 17, 11:59pm

Question 1.1: A complete normed vector space V (that is, a Banach space) with norm $\|\cdot\|$ is *uniformly convex* if for all all $0 < \epsilon \le 2$, there exists a $\delta = \delta(\epsilon) > 0$ such that whenever vectors $x, y \in V$ with $\|x\| = \|y\| = 1$ satisfy

$$\|x - y\| \ge \epsilon, \quad \text{then} \quad \left\| \frac{x + y}{2} \right\| \le 1 - \delta.$$

(a) Let V be a uniformly convex Banach space (so that V is complete) and $C \subset V$ be a closed convex set. Show that

$$\pi_C(x) := \operatorname*{argmin}_{y \in C} \|y - x\|$$

exists and is unique. *Hint.* Without loss of generality, you may assume $x = 0$ and $0 \notin C$.

(b) Let $\|\cdot\| = \|\cdot\|_1$ be the ℓ_1-norm. Show that $\mathbb{R}^n$ with this norm is not uniformly convex by giving a convex set C and point x for which $\pi_C(x)$ is not unique. Draw a picture.

(c) Let $\|\cdot\| = \|\cdot\|_\infty$ be the ℓ_∞-norm. Show that $\mathbb{R}^n$ with this norm is not uniformly convex by giving a convex set C and point x for which $\pi_C(x)$ is not unique. Draw a picture.