EE364M Exercise Set 3

Due date: January 31, 11:59pm

Question 3.1 (Inequalities with SDP Relaxations)**:** Let $A \in \mathcal{S}_n = \{X \in \mathbb{R}^{n \times n} \mid X = X^T\}$ be a symmetric matrix and consider the quantities

$$f^\star(A) := \max_{x \in \{-1,1\}^n} x^T A x \quad \text{and} \quad f_\star(A) := \min_{x \in \{-1,1\}^n} x^T A x \tag{3.1}$$

along with their semidefinite relaxations

$$s^\star(A) := \sup_{X \succeq 0} \{\mathrm{tr}(AX) \mid \mathrm{diag}(X) = \mathbf{1}\} \quad \text{and} \quad s_\star(A) := \inf_{X \succeq 0} \{\mathrm{tr}(AX) \mid \mathrm{diag}(X) = \mathbf{1}\}, \tag{3.2}$$

which satisfy $s_\star(A) \le f_\star(A) \le f^\star(A) \le s^\star(A)$. Let $\mathrm{sign}(x) = 1$ if $x \ge 0$ and $\mathrm{sign}(x) = -1$ otherwise, and for vectors $v \in \mathbb{R}^n$, define the elementwise sign mapping $\sigma(v) = (\mathrm{sign}(v_1), \ldots, \mathrm{sign}(v_n))$.

(a) Show that

$$f^\star(A) = \sup_{V \in \mathbb{R}^{n \times n}, u \in \mathbb{R}^n} \left\{\sigma(V^T u)^T A \sigma(V^T u) \mid \mathrm{diag}(V^T V) = \mathbf{1},\ \|u\|_2 = 1\right\}.$$

(An analogous result of course holds for $f_\star$.)

(b) Let $u \sim \mathsf{Uni}(\mathbb{S}^{n-1})$ have uniform distribution on the sphere. Show that

$$f^\star(A) = \sup_{V \in \mathbb{R}^{n \times n}} \left\{\mathbb{E}\left[\sigma(V^T u)^T A \sigma(V^T u)\right] \mid \mathrm{diag}(V^T V) = \mathbf{1}\right\}.$$

Hint. What happens when V is rank 1?

For the remainder of the problem, for a function $h : \mathbb{R} \to \mathbb{R}$ and matrix $X \in \mathbb{R}^{n \times n}$, let $h[X] \in \mathbb{R}^{n \times n}$ be the entrywise application of h to X, that is, the matrix whose (i, j) entry is $h(X_{ij})$.

(c) Show that

$$f^\star(A) = \frac{2}{\pi} \sup_{X \succeq 0, \mathrm{diag}(X) = \mathbf{1}} \mathrm{tr}(A \sin^{-1}[X]).$$

(d) Use that $\sin^{-1}[X] \succeq X$ whenever $|X_{ij}| \le 1$ for all entries i, j of X to show that for any diagonal matrix $D = \mathrm{diag}(d)$, $d \in \mathbb{R}^n$, with $A + D \succeq 0$, we have

$$f^\star(A) \ge \frac{2}{\pi} \sup_{X \succeq 0, \mathrm{diag}(X) = \mathbf{1}} \mathrm{tr}(AX) - \left(1 - \frac{2}{\pi}\right) \mathbf{1}^T d.$$

(See Question 3.2 below for the easy(ish) proof of the arcsin inequality.)

(e) Here we preview what is to come a bit in the class. Using the duality theory we develop, we may show that

$$s_\star(A) = \inf_{X \succeq 0} \{\mathrm{tr}(AX) \mid \mathrm{diag}(X) = \mathbf{1}\} = \sup_{d \in \mathbb{R}^n} \left\{-d^T \mathbf{1} \mid A + \mathrm{diag}(d) \succeq 0\right\}.$$

Conclude that we have the approximation bounds

$$s^\star(A) \ge f^\star(A) \ge \frac{2}{\pi} s^\star(A) + \left(1 - \frac{2}{\pi}\right) s_\star(A).$$

An additional quite optional question (for flavor)

Question 3.2 (Spectra of Kronecker and Hadamard products)**:** The Kronecker product $C = A \otimes B$ of matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is the matrix $C \in \mathbb{R}^{pm\times qn}$ with block entries

$$C = \begin{bmatrix} A_{11}B & A_{12}B & \cdots & A_{1n}B \\ A_{21}B & A_{22}B & \cdots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \cdots & A_{mn}B \end{bmatrix}.$$

(a) Let $A \in \mathcal{S}_m$ and $B \in \mathcal{S}_n$. Give the eigenvalues and eigenvectors of $A \otimes B$ in terms of those of A and B.

(b) Using part (a), show that the Hadamard (elementwise) product of positive semidefinite matrices is positive semidefinite, that is, if $A, B \succeq 0$, then $A \odot B \succeq 0$.

For the remainder of the problem, for a function $h : \mathbb{R} \to \mathbb{R}$ and matrix $X \in \mathbb{R}^{n\times n}$, let $h[X] \in \mathbb{R}^{n\times n}$ be the entrywise application of h to X, that is, the matrix whose (i, j) entry is $h(X_{ij})$.

(c) Let $X \succeq 0$. Show that $[X]^k \succeq 0$ for all $k \in \mathbb{N}$.

(d) Let $X \succeq 0$. Show that if $|X_{ij}| \le 1$ for all i, j, then $\sin^{-1}[X] \succeq X$. *Hint.* Use the (infinite) Taylor expansion of $\sin^{-1}$, which has all positive coefficients.