EE364M Exercise Set 4

Due date: February 7, 11:59pm

Question 4.1 (Strong duality with conic inequalities)**:** Let $\mathcal{H}$ be a Hilbert space (complete normed vector space with inner product $\langle \cdot, \cdot \rangle$) and $K \subset \mathcal{H}$ be a convex cone[1] with non-empty interior. We say

$$x \succeq 0 \text{ if } x \in K \quad \text{and} \quad x \succ 0 \text{ if } x \in \operatorname{int} K.$$

The dual cone associated with K is then

$$K^* := \{v \mid \langle v, x \rangle \geq 0 \text{ for all } x \in K\}.$$

Let $\mathcal{X}$ also be a Hilbert space. A mapping $G : \mathcal{X} \to \mathcal{H}$ is K-convex if its domain is convex and

$$G(tx + (1-t)y) \preceq tG(x) + (1-t)G(y) \text{ for all } t \in [0,1], \; x, y \in \operatorname{dom} G.$$

Let $H : \mathcal{X} \to \mathbb{R}^k$ be an affine function. Consider the convex problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & G(x) \preceq 0, \; H(x) = 0, \; x \in \Omega \end{array} \tag{4.1}$$

where $\Omega \subset \mathcal{X}$ is convex and both $\Omega \subset \operatorname{dom} f$ and $\Omega \subset \operatorname{dom} G$, with optimal value $\omega^\star$. Assume the following constraint qualification: there exists $x_0 \in \Omega$ such that

$$G(x_0) \prec 0 \quad \text{and} \quad H(x_0) = 0, \tag{Slater}$$

and additionally that $0 \in \operatorname{int}\{y \in \mathbb{R}^k \mid H(x) = y \text{ for some } x \in \Omega\}$.

Show that strong duality obtains for problem (4.1), that is, there exist

$$\lambda^\star \in K^* \quad \text{and} \quad \nu^\star \in \mathbb{R}^k \quad \text{such that} \quad \inf_{x \in \Omega} \{f(x) + \langle \lambda^\star, G(x) \rangle + \langle \nu^\star, H(x) \rangle\} = \omega^\star.$$

Hints. You may use the following form of the separating hyperplane theorem.

Proposition 4.1.1 (Eidelheit Separation)**.** *Let A, B be convex sets in a Hilbert space $\mathcal{X}$ such that A has non-empty interior and $\operatorname{int} A \cap B = \emptyset$. Then there exists $\lambda \in \mathcal{X}$, $\lambda \neq 0$, such that*

$$\inf_{a \in A} \langle \lambda, a \rangle \geq \sup_{b \in B} \langle \lambda, b \rangle.$$

It will be useful to prove that $A := \{(u, y, t) \mid f(x) \leq t, G(x) \preceq u, H(x) = y \text{ for some } x \in \Omega\}$ has non-empty interior. This is not a completely trivial statement. You should feel free to assume that the Hilbert spaces are finite-dimensional (i.e., $\mathbb{R}^n$). I have two solutions; each uses one of the Lemmas 4.1.2 or 4.1.3, both of which follow by combining Proposition 4.1.1 with the following openness guarantee for convex sets.

Lemma 4.1.1 (Hiriart-Urruty and Lemaréchal [1], Lemma III.2.1.6)**.** *Let C be a convex set in a vector space with norm $\|\cdot\|$. If $x \in \operatorname{int} C$ and $y \in \operatorname{cl} C$, then the half-open segment $[x, y) := \{(1-t)x + ty \mid 0 \leq t < 1\} \subset \operatorname{int} C$.*

[1] For this question, we say that K is a cone if for any $x \in K$, $tx \in K$ for $t > 0$. We do not require that $0 \in K$.

The book provides the result for $C \subset \mathbb{R}^n$, but the proof extends to any vector space.

If you use one of the following results, you should prove it.

Lemma 4.1.2. *Let K be a convex cone in a Hilbert space $\mathcal{H}$ with non-empty interior. Then the following hold.*

i. *Let $u \in \operatorname{int} K$. For $\lambda \in K^*$, $\langle \lambda, u \rangle = 0$ if and only if $\lambda = 0$. In particular, any vector $\lambda \in K^* \cap -K^*$ is zero.*

ii. *$\operatorname{int} K$ is a convex cone.*

iii. *For any two vectors $x_0, x_1 \in \mathcal{H}$, $(x_0 + \operatorname{int} K) \cap (x_1 + \operatorname{int} K)$ is non-empty.*

Lemma 4.1.3. *Let K be a convex cone in a Hilbert space $\mathcal{H}$. Then the following hold.*

i. *Let $u \in \operatorname{int} K$. Then for $\lambda \in K^*$, $\langle \lambda, u \rangle = 0$ if and only if $\lambda = 0$.*

ii. *$\operatorname{cl} K = K^{**}$, and so if K is closed, then $K = K^{**}$.*

iii. *Define $\gamma_{\min}(u) := \inf_{\lambda \in K^*} \{\langle \lambda, u \rangle \mid \|\lambda\| = 1\}$. Then $\gamma_{\min}(u) \geq 0$ if and only if $u \in \operatorname{cl} K$. If $\operatorname{int} K \neq \emptyset$, then $\gamma_{\min}(u) > 0$ if and only if $u \succ 0$.*

References

[1] J. Hiriart-Urruty and C. Lemaréchal. *Convex Analysis and Minimization Algorithms I*. Springer, New York, 1993.