ee364m Exercise Set 5

Due date: February 14, 11:59pm

Question 5.1 (The Prékopa-Leindler inequality)**:** The Prékopa-Leindler inequality states that if $f, m, g : \mathbb{R}^n \to \mathbb{R}_+$ are nonnegative functions satisfying

$$m(\lambda x + (1-\lambda)y) \ge f(x)^\lambda g(y)^{1-\lambda}$$

for all $x, y \in \mathbb{R}^n$ and $\lambda \in (0,1)$, then

$$\int m(x)dx \ge \left(\int f(x)dx\right)^\lambda \left(\int g(y)dy\right)^{1-\lambda}. \tag{PL}$$

In class, we proved inequality (PL) for the case $n = 1$. Perform the induction to demonstrate that it holds for general n.

Question 5.2 (Statistical(ish) consequences of Prékopa-Leindler)**:**

(a) Use the Prékopa-Leindler inequality to show that if $f(x,y)$ is (jointly) log-concave in (x,y), then the partial integration

$$F(x) := \int f(x,y)dy$$

is log-concave in x.

(b) A random variable X with density f has *log-concave density* if $\log f$ is concave. Show that if X and Y are independent random variables with log concave densities, then $Z = X + Y$ has log concave density.

(c) *Anderson's inequality* addresses optimally estimating the location of a random variable. Let $Z \in \mathbb{R}^n$ be a random variable with log-concave and symmetric density f, and $L : \mathbb{R}^n \to \mathbb{R}_+$ be any symmetric quasi-convex loss. Show that

$$\inf_{x \in \mathbb{R}^n} \mathbb{E}[L(x+Z)] = \mathbb{E}[L(Z)].$$