EE364M Exercise Set 6

Due date: February 21, 11:59pm

Question 6.1 (Cutting plane methods and the center of gravity)**:** Cutting plane methods solve convex optimization problems by iteratively constructing a polyhedron P_k guaranteed to contain the minimizer, at each step "cutting" part of the polyhedron. The basic method is as follows: consider the problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & x \in P_0 \end{array} \tag{6.1}$$

where $P_0 \subset \mathbb{R}^n$ is some compact convex body and f is subdifferentiable on P_0, so that the minimizer $x^\star \in P_0$ exists. Then at iteration k (for $k = 0, 1, \ldots$), we have some current localization set P_k (typically a polyhedron), choose $x_k \in P_k$. To construct a new localization set P_{k+1}, we select some $g_k \in \partial f(x_k)$, and observe that

$$f(x^\star) \ge f(x_k) + g_k^T(x^\star - x_k),$$

and in particular, $x^\star \in \{y \in \mathbb{R}^n \mid g_k^T(y - x_k) \le 0\}$. Thus, so long as $x^\star \in P_k$, it must also belong to the new *localization set*

$$P_{k+1} := P_k \cap \left\{y \in \mathbb{R}^n \mid g_k^T(y - x_k) \le 0\right\}.$$

We then take

$$\widehat{x}_k := \operatorname*{argmin}_{x \in \{x_0, \ldots, x_k\}} f(x)$$

to be the iterate with smallest objective value. (See, for example, Boyd and Vandenberghe's notes on localization methods.) The choice of $x_k \in P_k$ is an essential part of the convergence of this method; later, we will consider using the center of gravity as this choice.

Define the relative optimality gap of a point $x \in P_0$ by

$$\mathsf{gap}(x) := \frac{f(x) - f^\star}{\sup_{x \in P_0} f(x) - f^\star},$$

where $f^\star = \inf_{x \in P_0} f(x)$. (Note that $\sup_{x \in P_0} f(x) < \infty$ as P_0 is compact and f is continuous.) We will demonstrate that $\mathsf{gap}(\widehat{x}_k)$ converges linearly to zero when we choose x_k to be the center of gravity. We begin with generalities about cutting plane methods. Let size be a positively homogeneous and shift-invariant measure of a set's size, meaning that

$$\text{size}(\alpha P) = \alpha\, \text{size}(P) \quad \text{and} \quad \text{size}(x + P) = \text{size}(P) \quad \text{for } \alpha \ge 0,\ x \in \mathbb{R}^n,$$

that $\text{size}(P) > 0$ if P has non-empty interior, and $\text{size}(P_0) \ge \text{size}(P_1)$ if $P_0 \supset P_1$.

(a) Argue in a sentence or less that $\text{diam}(P) = \sup_{x,y \in P} \|x - y\|$ and $\text{Vol}_n(P)^{1/n}$ are size measures.

(b) Show that if $y \in P_0 \setminus P_k$, then $f(y) > f(\widehat{x}_k)$.

(c) Let $x^\star$ solve problem (6.1) and fix $0 < \alpha \le 1$. Define

$$P^\alpha := x^\star + \alpha(P_0 - x^\star) = (1 - \alpha)x^\star + \alpha P_0.$$

Show that if $y \in P^\alpha$, then the relative optimality gap satisfies

$$\mathsf{gap}(y) \le \alpha.$$

(d) Show that if $1 \geq \alpha > \text{size}(P_k)/\,\text{size}(P_0)$, then $P^\alpha \setminus P_k$ is non-empty.

(e) Show that for any cutting plane method and any such size measure,

$$\mathsf{gap}(\widehat{x}_k) \leq \frac{\text{size}(P_k)}{\text{size}(P_0)}.$$

For the remainder of the problem, consider Levin's center of gravity cutting plane method, which chooses

$$x_k = \text{cg}(P_k) := \frac{1}{\text{Vol}_n(P_k)} \int_{P_k} x\,dx. \tag{6.2}$$

(f) Show that the center-of-gravity method satisfies

$$\mathsf{gap}(\widehat{x}_k) \leq \left(1 - \frac{1}{e}\right)^{\frac{k}{n}}$$

and consequently, given subgradient access to a convex function $f : \mathbb{R}^n \to \mathbb{R}$, one can in principle minimize it to (relative) accuracy ε in $O(1) \cdot n \log \frac{1}{\varepsilon}$ subgradient calculations.