EE364M Exercise Set 7

Due date: February 28, 11:59pm

Question 7.1 (An efficient representation of robustness)**:** In robust optimization problems, one has uncertain problem data: abstractly, there is an uncertainty set $\mathcal{U}$ capturing potential noise, and we wish to guarantee that problem constraints are satisfied for all potential values $u \in \mathcal{U}$. This (abstractly) gives rise to convex optimization problems of the form

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & g_i(x, u) \le 0 \text{ for all } u \in \mathcal{U}, \quad i = 1, \ldots, m, \end{array} \tag{7.1}$$

where for each $u \in \mathcal{U}$, $g_i(\cdot, u)$ is convex. See, for example, the book [1] for much much more on such problems. The challenge is that, while the problem (7.1) is convex whenever f is, the constraints in problem (7.1) are typically infinite, making efficient computation challenging; a common approach is to use duality to obtain an alternative representation of the constraint $g_i(x, u) \le 0$ for all $u \in \mathcal{U}$ that is efficiently representable. Typical choices for $\mathcal{U}$ include sets defined by probability distributions, so that $\mathcal{U}$ covers (say) a $1 - \alpha$ fraction of possible realizations of noise u in a system.

Recall the ℓ_2-operator norm $\|\|A\|\|_{\text{op}} = \sup\{u^T A v \mid \|u\|_2 = \|v\|_2 = 1\}$. Fix (nominal) problem data A_0, b, let $\gamma \ge 0$, and consider the constraint that $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$ satisfy

$$\|(A_0 + \Delta)x + b\|_2 \le t \ \text{ for all } \ \|\|\Delta\|\|_{\text{op}} \le \gamma. \tag{7.2}$$

Show that (x, t) satisfy the constraint (7.2) if and only if there exists $\lambda \ge 0$ such that

$$\begin{bmatrix} tI_n - \lambda I_n & 0 & (A_0 x + b) \\ 0 & \lambda I_n & \gamma x \\ (A_0 x + b)^T & \gamma x^T & t \end{bmatrix} \succeq 0.$$

Hints. In my solution, I used the following results. If you use Lemma 7.1.1, you should prove it. You may assume Lemmas 7.1.2 and 7.1.3.

Lemma 7.1.1. *Let $K = \{(x, t) \mid \|x\|_2 \le t\}$ be the Lorentz cone. Then $(x, t) \in K$ if and only if*

$$\begin{bmatrix} tI & x \\ x^T & t \end{bmatrix} \succeq 0.$$

Lemma 7.1.2 (Schur complements)**.** *Let A, B, C be matrices of appropriate size. Then*

$$M = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \succeq 0$$

if and only if $A \succeq 0$, $C - B^T A^\dagger B \succeq 0$, and $\text{span}(A)$ *contains the columns of B.*

Using these, I showed that the constraint (7.2) was equivalent to

$$t\|v\|_2^2 + ts^2 + 2sv^T(A_0 x + b) - 2\gamma s u^T x \ge 0 \ \text{ whenever } \ \|v\|_2^2 \ge \|u\|_2^2, \ s \in \mathbb{R}. \tag{7.3}$$

Now apply

Lemma 7.1.3 (Inhomogeneous S-Lemma)**.** *Assume the constraint qualification that there exists* $\overline{x}$ *such that* $\overline{x}^T A_0 \overline{x} + 2b_0^T \overline{x} + c_0 > 0$*. Then the following two statements are equivalent:*

$$x^T A_0 x + 2b_0^T x + c_0 \geq 0 \quad \textit{implies} \quad x^T A_1 x + 2b_1^T x + c_1 \geq 0$$

and

$$\begin{bmatrix} A_1 - \lambda A_0 & b_1 - \lambda b_0 \\ (b_1 - \lambda b_0)^T & c_1 - \lambda c_0 \end{bmatrix} \succeq 0 \quad \textit{for some } \lambda \geq 0.$$

References

[1] A. Ben-Tal, L. E. Ghaoui, and A. Nemirovski. *Robust Optimization.* Princeton University Press, 2009.