EE364M Exercise Set 8

Due date: March 6, 11:59pm

In this exercise set, we will develop some of the theory of logarithmically homogeneous self-concordant functions. This can require some care with regards to the boundaries of domains of self-concordant functions, and to that end, we make a slightly extended definition of self concordance.

Definition 8.1. *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be thrice differentiable on its domain $\Omega = \operatorname{dom} f$. Then f is (strongly non-degenerate) self-concordant if*

i. it is strictly convex on Ω (equivalently, $\nabla^2 f(x) \succ 0$ for $x \in \Omega$),

ii. for each $x \in \Omega$ and $v \in \mathbb{R}^n$, the functional $\phi(t) := f(x + tv)$ satisfies

$$\phi'''(t) \le 2(\phi''(t))^{3/2},$$

iii. and $f(x) \to \infty$ as $x \to \operatorname{bd} \Omega$.

Strong non-degeneracy corresponds to strict convexity of f and boundary growth; we assume all self-concordant functions we work with satisfy this condition without further comment.

Question 8.1: Let $K \subset \mathbb{R}^n$ be a convex cone with interior, and let $f : \operatorname{int} K \to \mathbb{R}$ be self-concordant with domain $\operatorname{dom} f = \operatorname{int} K$. Define the Hessian and gradient mappings $g(x) = \nabla f(x)$ and $H(x) = \nabla^2 f(x)$. We say that f is and ν-logarithmically homogeneous if

$$f(tx) = f(x) - \nu \log t$$

for $t > 0$.

(a) Show that if f is logarithmically homogeneous, then

$$g(tx) = \frac{1}{t} g(x) \quad \text{for all } t > 0. \tag{8.1}$$

(b) Show that if f has gradient satisfying the identity (8.1), then

$$H(tx) = \frac{1}{t^2} H(x) \quad \text{and} \quad H(x)x = -g(x).$$

Use these to show that that the (squared) Newton decrement

$$\lambda^2(x) := \langle \nabla f(x), \nabla^2 f(x)^{-1} \nabla f(x) \rangle,$$

is constant and equals $\lambda^2(x) = -\langle g(x), x \rangle$.

(c) Show that if $f : \operatorname{int} K \to \mathbb{R}$ has gradient satisfying the identity (8.1), then

$$f(tx) = f(x) - \nu \log t,$$

where $\nu = \sup_{x \in \operatorname{dom} f} \lambda^2(x)$. Use this to conclude that f is ν-logarithmically homogeneous if and only if identity (8.1) holds, and in this case, $\nu = \lambda^2(x)$ for any x.

For the remainder of the question, assume that $f : \operatorname{int} K \to \mathbb{R}$ is ν-logarithmically homogeneous.

(d) Show that $\nabla f(x) \in -K^*$ for any $x \in K$.

(e) Use Lemma 4.1.3 from Exercise 4.1 to argue that for any $\lambda \in -\operatorname{int} K^*$, there exists $\alpha > 0$ such that $\langle -\lambda, x \rangle \geq \alpha \|x\|$ for $x \in K$. Then show that if $h(t) := f(tx) - \langle \lambda, tx \rangle$, we obtain $h'(t) > 0$ whenever $t > 1$ and $\|x\| > \nu/\alpha$. Conclude that

$$f(x) - \langle \lambda, x \rangle$$

has a minimizer satisfying $\|x\| \leq \nu/\alpha$.

(f) Use the previous parts and Lemma 8.1.2 (to follow) to show that $\operatorname{dom} f^* = -\operatorname{int} K^*$.

Note that you have shown the following theorem:

Theorem 8.1.1. *Let f be self-concordant and ν-logarithmically homogeneous with domain* $\operatorname{dom} f = \operatorname{int} K$*. Then f^* is self-concordant, ν-logarithmically homogeneous, and* $\operatorname{dom} f^* = -\operatorname{int} K^*$*.*

(The logarithmic homogeneity of f^* follows by the identity $\nabla^2 f(x)^{-1} = \nabla^2 f^*(s)$ when $\nabla f(x) = s$.)

The following lemmas may be useful for answering the question.

Lemma 8.1.1 (Boyd and Vandenberghe [1], Ex. 9.19)**.** *Let f be self concordant and assume that $\inf_x f(x) > -\infty$. Then f has a minimizer.*

Lemma 8.1.2. *Let f be self concordant with domain $\Omega = \operatorname{dom} f$. Then Ω is open, the gradient mapping $g(x) := \nabla f(x)$ is injective, and* $\operatorname{dom} f^* = \{g(x) \mid x \in \Omega\}$ *and is open.*

Proof That Ω is open follows because

$$\operatorname{dom} f = f^{-1}(\mathbb{R}) = f^{-1}(-\infty, \infty)$$

and f is continuous. To show injectivity of g, suppose $g(x_0) = s$ and $g(x_1) = s$. Then x_0 and x_1 both minimize $f(x) - \langle s, x \rangle$ by the Fenchel-Young inequality; this is strictly convex and hence $x_0 = x_1$. So g is injective (one-to-one). Then again by the equality condition in the Fenchel-Young inequality, we have $x = \nabla f^*(s)$ and $f^*(s) = \langle x, s \rangle - f(x) < \infty$ so that $s \in \operatorname{dom} f^*$. That is,

$$\{g(x) \mid x \in \Omega\} \subset \operatorname{dom} f^*.$$

We now show that $\operatorname{dom} f^* \subset \{g(x) \mid x \in \Omega\}$. Let $s \in \operatorname{dom} f^*$, so that $\sup_x \langle s, x \rangle - f(x) < \infty$. Then $\inf_x f(x) - \langle s, x \rangle > -\infty$, so that the self-concordant function $x \mapsto f(x) - \langle s, x \rangle$ has a minimizer (Lemma 8.1.1). Then there exists an x such that $\nabla f(x) - s = 0$, i.e., $g(x) = s$. □

References

[1] S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, 2004.