EE364M Exercise Set 9

Due date: March 13, 11:59pm

Question 9.1 (Properties of log-concave distributions)**:** A probability distribution P on $\mathbb{R}^n$ is *log concave* if

$$P(\lambda A + (1-\lambda)B) \geq P(A)^{\lambda} P(B)^{1-\lambda}$$

for all (measurable) $A, B \subset \mathbb{R}^n$ and $\lambda \in [0,1]$.

(a) Let P have density $p(x) = e^{-u(x)}$ where $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is convex. Use the Prékopa-Leindler inequality to show that P is log-concave.

(b) Let P be a log-concave probability distribution on $\mathbb{R}^n$ with continuous density p. Show that $p(x) = e^{-u(x)}$ for a convex $u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$.

(c) Let P be a log-concave probability distribution. Show that for any convex symmetric $A \subset \mathbb{R}^n$ and $r \geq 1$ that

$$1 - P(rA) \leq \left(\frac{1-P(A)}{P(A)}\right)^{r/2} \sqrt{P(A)(1-P(A))},$$

where $rA = \{rx \mid x \in A\}$. *Hint.* First, show that if $y \notin rA$ and $x \in A$, then $z = \frac{2}{r+1}y + \frac{r-1}{r+1}x$ must be outside A. Conclude that

$$A^c \supset \frac{2}{r+1}(rA)^c + \frac{r-1}{r+1}A$$

and then apply log-concavity.

As a comment, the last problem shows that if $P(A)$ is reasonably large—even bigger than $\frac{1}{2}$—then rA covers an exponentially large portion of the probability mass of P. For example, if $P(A) \geq \frac{e}{e+1} \approx .731$, then $P(rA) \geq 1 - e^{-r/2}\sqrt{e}/(e+1) > 1 - \frac{1}{2}e^{-r/2}$.